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LETTER TO THE EDITOR

Group theory approach to relativistic scattering

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Abstract. An algebraic technique, useful in studying relativistic scattering of a Dirac particle in a certain class of external fields, is presented. As an example, this technique is applied to the algebraic determination of the S-matrix for Coulomb–Dirac scattering. The technique requires the introduction of linear invariant operators or linear invariant coset operators (LCO).

Algebraic techniques, useful in a variety of bound state problems in physics (for a review, see [1] and references therein), have recently been extended to non-relativistic scattering problems [2, 3]. The relevant groups are non-compact and scattering states are described by their continuous unitary representations. For systems with dynamic symmetry, the functional form of the S-matrix is determined by an expansion formula which connects the asymptotic generators of the dynamic algebra to those of a pseudo-Euclidean algebra. Realistic models for heavy-ion scattering based on such S-matrices have been constructed [4].

Relativistic scattering problems by group theory have been discussed extensively in the past but mostly in the context of infinite-component wave equations (see, for example, [5]).

The purpose of this paper is to show that our algebraic methods for non-relativistic scattering can be extended to *relativistic* scattering problems which are described by standard finite-component wave equations. In particular, we discuss here the scattering of a Dirac particle in an external field. Equations of this type have been applied recently to the study of the scattering of intermediate energy protons off nuclei [6] (Dirac phenomenology). Although we discuss here only scattering in a Coulomb field (in three spatial dimensions), we emphasise the fact that the technique presented here can also be applied to other solvable Dirac problems and that, in general, it can form the basis for realistic models of relativistic collisions.

A group theoretical approach to the Dirac–Coulomb problem has been worked out by several authors [7–10] but this has been done by converting the Dirac equation into a second-order differential equation. Since one of the major new ingredients of the Dirac equation is that the differential equation is of first order in the spatial variables, we want to present here a new group theoretic technique which attacks directly the equation at the first-order level.

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In the group theoretic approach to non-relativistic scattering [1-4], we have exploited the properties of quadratic Casimir invariants. This is because the Schrödinger equation is of second order in the spatial derivatives. We suggest that in treating the Dirac equation one should introduce *linear* invariants which necessarily imply the use of matrix representations and Clifford algebras [11]. We are then naturally led to a set of coupled equations. The solution is obtained by an algebraic treatment of this coupled-channel problem.

We begin with a brief discussion of the construction of a linear Casimir operator. Suppose that $\mathscr{G}$ is a Lie algebra defined through the structure constants f^{ij}_k;

$$[F^i, F^j] = f^{ij}_k F^k. \tag{1}$$

Let ϕ^i be a finite matrix representation of the same algebra such that $[\phi^i, F^j] = 0$. We can then construct a 'coupled' representation of the algebra $\Phi^i = \phi^i + F^i$. A trivial example is provided by $\mathscr{G} = \mathrm{SU}(2)$ with F^i, ϕ^i and Φ^i corresponding to the orbital, spin and total angular momentum, respectively. The linear Casimir operator is defined by

$$L(\phi, F) = \{C(\Phi) - C(F) - C(\phi)\}/2 = \phi_i F^i \equiv \phi \cdot F \tag{2}$$

where $C(\phi) = \phi_i\phi^i$, etc, are quadratic Casimir invariants and the index i is lowered by means of the Killing metric $g^{ij} = f^{ik}_l f^{jl}_k$. Note that although we have used two copies of $\mathscr{G}$ (associated with $\mathscr{G}\otimes\mathscr{G}$) the linear Casimir operator is invariant only under $\mathscr{G}$, the diagonal subgroup of $\mathscr{G}\otimes\mathscr{G}$. A Dirac-like equation can be obtained from (2) by choosing a realisation of F in terms of first-order differential operators and for ϕ in terms of γ-matrices. This technique can be generalised, if one wishes, to the construction of invariant linear coset operators (LCO). This is obtained by considering a subalgebra $\mathscr{H}$. The algebra $\mathscr{G}$ can then be decomposed into $\mathscr{H}$ and $\mathscr{P} = \mathscr{G} - \mathscr{H}$ which corresponds to the coset $P = G/H$. One can then define a linear invariant LCO from $L(F_g, \phi_g) - L(F_h, \phi_h)$ where the subscripts g and h refer to the algebras $\mathscr{G}$ and $\mathscr{H}$.

In order to illustrate this general technique, we discuss here the Dirac equation of a charged particle of mass m in the presence of a potential $A_\mu = (V(r), 0, 0, 0)$,

$$[\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m + V(r)]\psi = \varepsilon\psi. \tag{3}$$

The matrices $\boldsymbol{\alpha}$, β are related to the Dirac matrices γ_μ in the usual way $\gamma_\mu = (\beta, \beta\boldsymbol{\alpha})$, and are constructed from two sets of Pauli Matrices $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$ by $\beta = \rho_3 \otimes 1$ and $\boldsymbol{\alpha} = \rho_1 \otimes \boldsymbol{\sigma}$. For arbitrary $V(r)$ this equation does not have any particular algebraic structure. However, we show in the paragraphs below that when $V(r) = \eta/r$ (Coulomb–Dirac equation) this equation has the structure of two coupled $\mathrm{SO}(2,1)\otimes\mathrm{SO}(3)$ algebras, where one of these pairs is a spatial realisation and the other is a matrix realisation. To this end, we first separate the radial part from the angular part. This corresponds to the separation

$$\begin{array}{ccccccc}(\mathrm{SO}(2,1) \otimes & \mathrm{SO}(3)) & \otimes & (\mathrm{SO}(2,1) \otimes & \mathrm{SO}(3)) \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ \text{radial part} & \text{orbital part} & & \rho\text{-matrices} & \sigma\text{-matrices(spin)}\end{array} \tag{4}$$

where we have denoted by SO(3) the spin group, although, properly speaking, it should be denoted by Spin(3). The orbital part and the spin are combined into a coupled representation

$$\begin{array}{ccccccc}\mathrm{SO}(3) & \otimes & \mathrm{SO}(3) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{L_1, L_2, L_3\} & & \{\sigma_1, \sigma_2, \sigma_3\} & & \{J_1, J_2, J_3\} & & \{J_3\}\end{array}. \tag{5}$$

This corresponds to the familiar addition of orbital $\boldsymbol{L}$, and spin, $\frac{1}{2}\boldsymbol{\sigma}$, angular momenta to yield the total angular momentum $\boldsymbol{J}=\boldsymbol{L}+\frac{1}{2}\boldsymbol{\sigma}$. In order to characterise the wavefunctions one needs here three quantum numbers in addition to the eigenvalues of J_3. These can be chosen by diagonalising three operators, for example $\boldsymbol{L}^2$, $\boldsymbol{\sigma}^2/4$ and $\boldsymbol{J}^2$. The eigenvalues of $\boldsymbol{\sigma}^2/4$ are trivially given by 3/4 and will not be considered further. The use of linear invariants implies that instead of L^2 we consider the invariant $\boldsymbol{\sigma}\cdot\boldsymbol{L}$. Indeed this is already what is usually done except that one diagonalises not $\boldsymbol{\sigma}\cdot\boldsymbol{L}$ but the operator $K=\beta(1+\boldsymbol{\sigma}\cdot\boldsymbol{L})$. Thus the angular problem is solved by considering simultaneous eigenstates of

$$\begin{aligned} &\boldsymbol{J}^2\psi_{\kappa j m_j}=j(j+1)\psi_{\kappa j m_j} \\ &J_z\psi_{\kappa j m_j}=m_j\psi_{\kappa j m_j} \qquad \kappa=\begin{cases} (j+\frac{1}{2}) & \text{for } l=j-\frac{1}{2} \\ -(j+\frac{1}{2}) & \text{for } l=j+\frac{1}{2}\end{cases} \\ &K\psi_{\kappa j m_j}=\kappa\psi_{\kappa j m_j}, \end{aligned} \tag{6}$$

The wavefunction ψ separates into an angular part and a radial part according to

$$\psi_{\kappa j m_j}=\begin{pmatrix} f(r)\mathcal{Y}_{ljm_j} \\ -\mathrm{i}g(r)(\boldsymbol{\sigma}\cdot\boldsymbol{r})\mathcal{Y}_{ljm_j}\end{pmatrix} \tag{7}$$

and the radial wavefunction $\phi=\binom{f}{g}$ satisfies the coupled 2×2 equations

$$\left[\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}\rho_3-m\rho_1-\mathrm{i}\left(\varepsilon-\frac{\eta}{r}\right)\rho_2\right]\phi=0. \tag{8}$$

We want to show that this equation has a group structure and therefore can be solved in a straightforward way using algebraic techniques. Indeed it is related to the linear invariants of the lattice of groups

$$\begin{array}{ccccc} & & & \mathrm{SO}(2,1) & \\ & & & \downarrow & \\ & & & \{N_1,N_2,N_3\} & \\ \mathrm{SO}(2,1) & \otimes & \mathrm{SO}(2,1) & \supset & \supset\mathrm{SO}(2) \\ \downarrow & & \downarrow & & \downarrow \\ \{M_1,M_2,M_3\} & & \{\mathrm{i}\rho_1,\mathrm{i}\rho_2,\rho_3\} & & \{N_3\} \\ & & & \mathrm{SO}(2)\otimes\mathrm{SO}(2) & \\ & & & \downarrow \qquad \downarrow & \\ & & & \{M_3\} \quad \{\rho_3\} & \end{array} \tag{9}$$

where $\boldsymbol{M}$ is a coordinate realisation of SO(2, 1). This technique of coupling an orbital realisation of an algebra with a finite realisation of the same algebra in terms of Dirac matrices has been employed in the past for the dynamical group SO(4, 2) to describe *infinite-component* wave equations [12].

The linear Casimir invariant of the coupled SO(2, 1) algebra is

$$\boldsymbol{\rho}\cdot\boldsymbol{M}=\mathrm{i}\rho_1M_1+\mathrm{i}\rho_2M_2+\rho_3M_3 \tag{10}$$

while that of the coupled SO(2) algebra is ρ_3M_3.

The algebraic treatment of the radial Dirac equation (8) is slightly more complicated than that of the angular part (6) and of the non-relativistic radial equation [13]. The reason is that (8) contains four parameters, κ, m, ε and η. The representations of the first chain of groups in (9) provide only three quantum numbers since the eigenvalues of $\boldsymbol{\rho}^2$ are trivial. In order to construct the most general radial Dirac equation we must use the invariants of all the groups appearing in (9). We therefore consider the simultaneous eigenvalues of

$$M^2\tilde{\Phi}_{\mu Mn} = M(M+1)\tilde{\Phi}_{\mu Mn} \tag{11a}$$

$$(\boldsymbol{\rho}\cdot\boldsymbol{M}+c\rho_3 M_3)\tilde{\Phi}_{\mu Mn} = \mu\tilde{\Phi}_{\mu Mn} \tag{11b}$$

$$N_3\tilde{\Phi}_{\mu Mn} = n\tilde{\Phi}_{\mu Mn}. \tag{11c}$$

We note that although $(\boldsymbol{\rho}\cdot\boldsymbol{M})$ and $(\rho_3 M_3)$ do not commute, each of them commutes with $\boldsymbol{M}^2$ and N_3 and thus their linear combination can be simultaneously diagonalised with $\boldsymbol{M}^2$ and N_3. Equation (11b) is related to the Dirac-Coulomb equation (8). In fact, consider the realisation of $\boldsymbol{M}$ in terms of two variables (r, θ),

$$\begin{aligned} M_\pm &= \frac{e^{\pm i\theta}}{2b}\left[-a \mp \left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial\theta}\right)\left(-2i\frac{\partial}{\partial\theta} \pm 1\right)\right] \\ M_3 &= -i\frac{\partial}{\partial\theta} \end{aligned} \tag{12}$$

with a, b constants. The eigenfunctions of $N_3 = M_3 + \rho_3/2$ have the form

$$\tilde{\Phi} = \begin{pmatrix} \tilde{f}(r)\exp[i(n-\frac{1}{2})\theta] \\ \tilde{g}(r)\exp[i(n+\frac{1}{2})\theta] \end{pmatrix}. \tag{13}$$

The θ dependence can be removed by a similarity transformation $R = \exp(i\theta\rho_3/2)$. After some manipulations (multiplication by $-b\rho_2/n$), one obtains an equation for $\tilde{\phi} = (\tilde{f}_{\tilde{g}})$,

$$\left[\frac{d}{dr} - \frac{n}{r}\rho_3 + \frac{a}{2n}\rho_3 - ib(c+1)\rho_1 + \frac{b}{n}\left(\mu + \frac{1}{2} + \frac{c}{2}\right)\rho_2\right]\tilde{\phi} = 0. \tag{14}$$

The Dirac-Coulomb equation (8) can be brought to this form by diagonalising [7] the matrix $\Lambda = \kappa\rho_3 + i\eta\rho_2$ (which we call the 'potential' matrix). This is simply achieved by the transformation

$$\exp(\chi\rho_1)\Lambda\exp(-\chi\rho_1) = \lambda\rho_3 \qquad \tanh 2\chi = \eta/\kappa \qquad \lambda = (\kappa^2-\eta^2)^{1/2}. \tag{15}$$

The transformed spinor $\tilde{\phi} = \exp(\chi\rho_1)\phi$ satisfies (14) with

$$a = 2\eta\varepsilon \qquad ib(c+1) = m \qquad n = \lambda \qquad b(\mu+\tfrac{1}{2}+\tfrac{1}{2}c) = i\varepsilon\kappa. \tag{16}$$

Once the group structure of the Dirac-Coulomb equation has been recognised, the computation of the associated S-matrix is straightforward and can be done by purely algebraic techniques. The S-matrix associated with (14) is diagonal. It is found from recursion relations derived algebraically using the 'Euclidean connection' techniques explained in [2, 4] where the incoming and outgoing asymptotic waves are viewed as different representations of the Euclidean group. The 'out' amplitudes are then given in terms of the 'in' amplitudes by

$$\tilde{\phi}^{\text{out}} = \begin{pmatrix} S_{-\lambda} & 0 \\ 0 & S_\lambda \end{pmatrix}\tilde{\phi}^{\text{in}} \tag{17}$$

with

$$S_\lambda = \exp(-i\pi\lambda)\Gamma(\lambda+1-i\eta\varepsilon/k)/\Gamma(\lambda+1+i\eta\varepsilon/k)$$

and

$$S_{-\lambda} = -[(\lambda+i\eta\varepsilon/k)/(\lambda-i\eta\varepsilon/k)]S_\lambda.$$

Here k is the magnitude of the momentum. The solution of the Dirac-Coulomb problem can be obtained from (17) by retransforming back the spinor $\tilde{\phi}$ to ϕ with

$$\exp(\chi\rho_1)=\begin{pmatrix}\cosh\chi & \sinh\chi\\ \sinh\chi & \cosh\chi\end{pmatrix}. \tag{18}$$

This allows one to determine the connection between ϕ^{out} and ϕ^{in}. One can then proceed to find the S-matrix, $S_{\kappa j}(k)=\exp(\mathrm{i}\pi l)f^{\mathrm{out}}/f^{\mathrm{in}}$, by eliminating in the usual way the small component g^{in} using the relation

$$g^{\mathrm{in}}=\frac{ik}{\varepsilon+m}f^{\mathrm{in}}. \tag{19}$$

One then obtains the standard form [14]

$$S_{\kappa j}(k)=\exp[-\mathrm{i}\pi(\lambda-l)]\left(\frac{-\kappa-\mathrm{i}\eta m/k}{\lambda-\mathrm{i}\eta\varepsilon/k}\right)\frac{\Gamma(\lambda+1-\mathrm{i}\eta\varepsilon/k)}{\Gamma(\lambda+1+\mathrm{i}\eta\varepsilon/k)}. \tag{20}$$

In conclusion: (i) the radial part of the Coulomb–Dirac equation is related to a matrix form of SO(2, 1) and it can be obtained from a combination of the linear invariants $\boldsymbol{\rho}\cdot\boldsymbol{M}$ and ρ_3M_3; (ii) the seemingly complex structure of the relativistic Coulomb–Dirac S-matrix is simply due to the rotation (18) and it can be disentangled by going to the diagonal form (17). Our analysis opens the way to several possibilities: (i) the construction and study of a class of solvable Dirac problems (for example the one-dimensional exponential potential, e^{-z}, and the potential step [15], tanh z); (ii) the construction of *realistic* models of relativistic scattering including the use of higher-dimensional algebras as in [4, 16]. A detailed discussion of these questions will be presented in a longer publication.

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